Physics 251

Final Project: Newton's Law of Gravitation – The Solar System

Newton's law of gravitation states that two celestial objects attract each other with a force that is proportional to the mass of each object and it is inversely proportional to the square of the distance between them (Feynman, Leighton, and Sands 1963; Giordano and Nakanishi 2006).

$$F = G\,\frac{m_1\, m_2}{r^2} \qquad 1$$

The Equation 1 describes the Newton's law of gravitation, where $G = 6.67430 \times 10^{-11}\, N \frac{m^2}{kg^2}$ is the gravitational constant, m_1 and m_2 are the masses of both objects, and $r = |\vec{r}|$ is the distance between the two objects, see Figure 1. The forces acting in each object are shown in Figure 2.

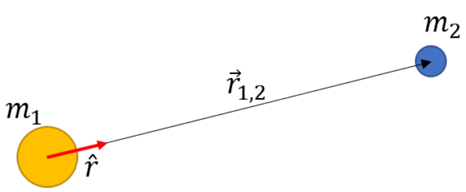

Figure 1: Vector distance between the centers of mass of two objects.

Figure 2: Attracting forces due to the Law of Gravitation.

Now, we can use Newton's second law, $\vec{F} = m\,\vec{a}$, to calculate the position of the object 1 and 2 as a function of time. If we assume $m_1 \gg m_2$, we can neglect the motion of the object 1, therefore, we can use the center of mass of object 1 as the origin of the coordinate system of the system of objects as is depicted in Figure 1.

The equation of motion of the object 2 is described by Newton's second law as follows:

$$\vec{F}_{2,1} = m_2\,\vec{a}_2$$

$$-G\,\frac{m_1\, m_2}{r_2^2}\,\hat{\boldsymbol{r}} = m_2\,\vec{a}_2 \qquad 2$$

where $\hat{\boldsymbol{r}}$ is the unitary vector pointing from the origin of the coordinate system, center of mass of object 1, towards to the center of mass of object 2, and $\vec{a}_2$ is the acceleration vector of the object 2. The vector $\vec{a}_2$ can be written in its Cartesian components:

$$\vec{a}_2 = a_{2_x}\,\hat{\boldsymbol{x}} + a_{2_y}\,\hat{\boldsymbol{y}} + a_{2_z}\,\hat{\boldsymbol{z}} \qquad 3$$

The unitary vector $\hat{\boldsymbol{r}}$ can be written as (Boas 2005; Taylor 2005; Griffiths 2017):

$$\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\,\hat{\boldsymbol{x}} + \sin\theta\sin\varphi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \qquad 4$$

where r, θ, and φ are the spherical coordinates.

Equation 2 can be rewritten using Equations 3 and 4, leading to the following set of three equations:

$$m_2 a_{2_x} = m_2 \frac{d^2 x_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^2} \sin\theta \cos\varphi \qquad 5$$

$$m_2 a_{2_y} = m_2 \frac{d^2 y_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^2} \sin\theta \sin\varphi \qquad 6$$

$$m_2 a_{2_z} = m_2 \frac{d^2 z_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^2} \cos\theta \qquad 7$$

If we multiply and divide by r the right hand side of Equations 5, 6, and 7, we obtain:

$$m_2 \frac{d^2 x_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^3}\, r_2 \sin\theta_2 \cos\varphi_2 \qquad 8$$

$$m_2 \frac{d^2 y_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^3}\, r_2 \sin\theta_2 \sin\varphi_2 \qquad 9$$

$$m_2 \frac{d^2 z_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^3}\, r_2 \cos\theta_2 \qquad 10$$

Now, we can rewrite Equations 8, 9, and 10 using the relation between the Cartesian coordinates and the spherical polar coordinates (Equations 21, 22, and 23 in the Appendix):

$$m_2 \frac{d^2 x_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^3}\, x_2 \qquad 11$$

$$m_2 \frac{d^2 y_2}{dt^2} = -G \frac{m_1\, m_2}{r_2^3}\, y_2 \qquad 12$$

$$m_2 \frac{d^2 z_2}{dt^2} = -G \frac{m_1 m_2}{r_2^3} z_2 \qquad 13$$

The spherical polar coordinate r can be written in terms of the Cartesian coordinates (see Equation 23 in the Appendix). Using r in terms of (x, y, z) in Equations 11,12, and 13, and then canceling m_2 in both sides of the equations, we obtain a set of three ordinary differential equations written in Cartesian coordinates that describes the motion of object 2:

$$\frac{d^2 x_2}{dt^2} = -G \frac{m_1}{\left(\sqrt{x_2^2 + y_2^2 + z_2^2}\right)^3} x_2 \qquad 14$$

$$\frac{d^2 y_2}{dt^2} = -G \frac{m_1}{\left(\sqrt{x_2^2 + y_2^2 + z_2^2}\right)^3} y_2 \qquad 15$$

$$\frac{d^2 z_2}{dt^2} = -G \frac{m_1}{\left(\sqrt{x_2^2 + y_2^2 + z_2^2}\right)^3} z_2 \qquad 16$$

Activities:

You must write a python script name sun_earh_3d.py to solve the equations of motion of the two body system in 3 dimensions, Sun-Earth, described by Equations 14, 15, and 16:

1. (30 points) Your script should have a function that solves the system of ODE's using Forward Euler. Name the function *solve_2body_fe:*
    a. the function should take as arguments: $t_0, t_n, \Delta t, \vec{x}_{Earth_0}$, and $\vec{v}_{Earth_0}$,
    b. and return the solution as an array t with all the times, and the arrays $\vec{x}_{Earth}$ and $\vec{v}_{Earth}$ with the positions and velocities you have calculated.
2. (30 points) Your script should have a function that solves the system of ODE's using Euler-Cromer. Name the function *solve_2body_ec*:
    a. the function should take as arguments: $t_0, t_n, \Delta t, \vec{x}_{Earth_0}$, and $\vec{v}_{Earth_0}$,
    b. and return the solution as an array t with all the times, and the arrays $\vec{x}_{Earth}$ and $\vec{v}_{Earth}$ with the positions and velocities you have calculated.
3. (40 points) Test *solve_2body_fe* and *solve_2body_ec* with the following inputs:
    a. $\Delta t = 0.1, 0.01, 0.001, 0.00001$ year:
    b. Initial conditions:
        i. Initial position:

$$x_{Earth_0} = 5.840342819515421 \times 10^{-01}\ AU$$
$$y_{Earth_0} = 7.980767254311480 \times 10^{-01}\ AU$$
$$z_{Earth_0} = -3.975661190069749 \times 10^{-05}\ AU$$

        ii. Initial velocity

$$v_{x_{Earth_0}} = -1.417095567994654 \times 10^{-02}\ AU/Day$$
$$v_{y_{Earth_0}} = 1.009895396220260 \times 10^{-02}\ AU/Day$$
$$v_{z_{Earth_0}} = 2.368804521251198 \times 10^{-07}\ AU/Day$$

    c. Time of integration:
        i. from $t_0 = 0$ to $t_n = 1.0$ year,
        ii. from $t_0 = 0$ to $t_n = 3.0$ years.
    d. Plot x_{Earth} vs t, y_{Earth} vs t, and z_{Earth} vs t for Forward Euler and Euler-Cromer integration methods and for all different possible inputs, e.g. Δt's and t_n's.
    e. Plot y_{Earth} vs x_{Earth}, z_{Earth} vs x_{Earth}, and z_{Earth} vs y_{Earth} for Forward Euler and Euler-Cromer integration methods and for all different possible inputs, e.g. Δt's and t_n's.
    f. Compare the Forward Euler and Euler-Cromer solutions.
        i. You must write a report with your findings. The report should have:
            1. Abstract.
            2. Introduction: physical model equations.
            3. Methods: numerical method equations.
            4. Results.
            5. Conclusion.
            6. References.
    g. HINT: Please be careful with the units. You must be consistent with the units you are using.

Appendix:

Spherical Polar Coordinates

The point P in Figure 1 can be written as (x, y, z) in the Cartesian coordinate system with origin at O. The spherical polar coordinate system is another possible representation of the point O (Boas 2005; Taylor 2005; Griffiths 2017), (r, θ, φ).

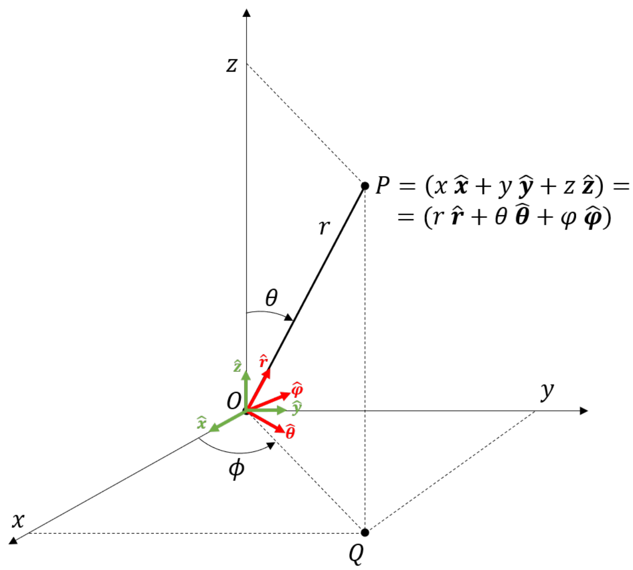

Figure 3: The spherical polar coordinates (r, θ, φ).

The unit vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ are mutually perpendicular, and they can be written in terms of the Cartesian unit vectors:

$$\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\ \hat{\boldsymbol{x}} + \sin\theta\sin\varphi\ \hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \qquad 17$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\ \hat{\boldsymbol{x}} + \cos\theta\sin\varphi\ \hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \qquad 18$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\ \hat{\boldsymbol{x}} + \cos\varphi\ \hat{\boldsymbol{y}} \qquad 19$$

The Cartesian coordinate (x, y, z) can be written in terms of the spherical polar coordinates as follows:

$$x = r\sin\theta\cos\varphi \qquad 20$$

$$y = r\sin\theta\sin\varphi \qquad 21$$

$$z = r\cos\theta \qquad 22$$

The spherical polar coordinates can be written in terms of the Cartesian coordinates as follows:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad 23$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \qquad 24$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) \qquad 25$$

References:

Boas, Mary L. 2005. *Mathematical Methods in the Physical Sciences*. 3rd ed.

Feynman, R. P., R.B. Leighton, and N. L. Sands. 1963. *The Feynman Lectures on Physics*. 3 vols. Reading, MA: Addison-Wesley Pub.Co. https://www.feynmanlectures.caltech.edu/I_toc.html.

Giordano, Nicholas J., and Hisao Nakanishi. 2006. *Computational Physics*. Second. Pearsons Education, Inc.

Griffiths, David J. 2017. *Introduction to Electrodynamics*. Fourth. Cambridge University Press.

Taylor, John R. 2005. *Classical Mechanics*. University Science Books.